MATH 312
Lecture 13: Crush

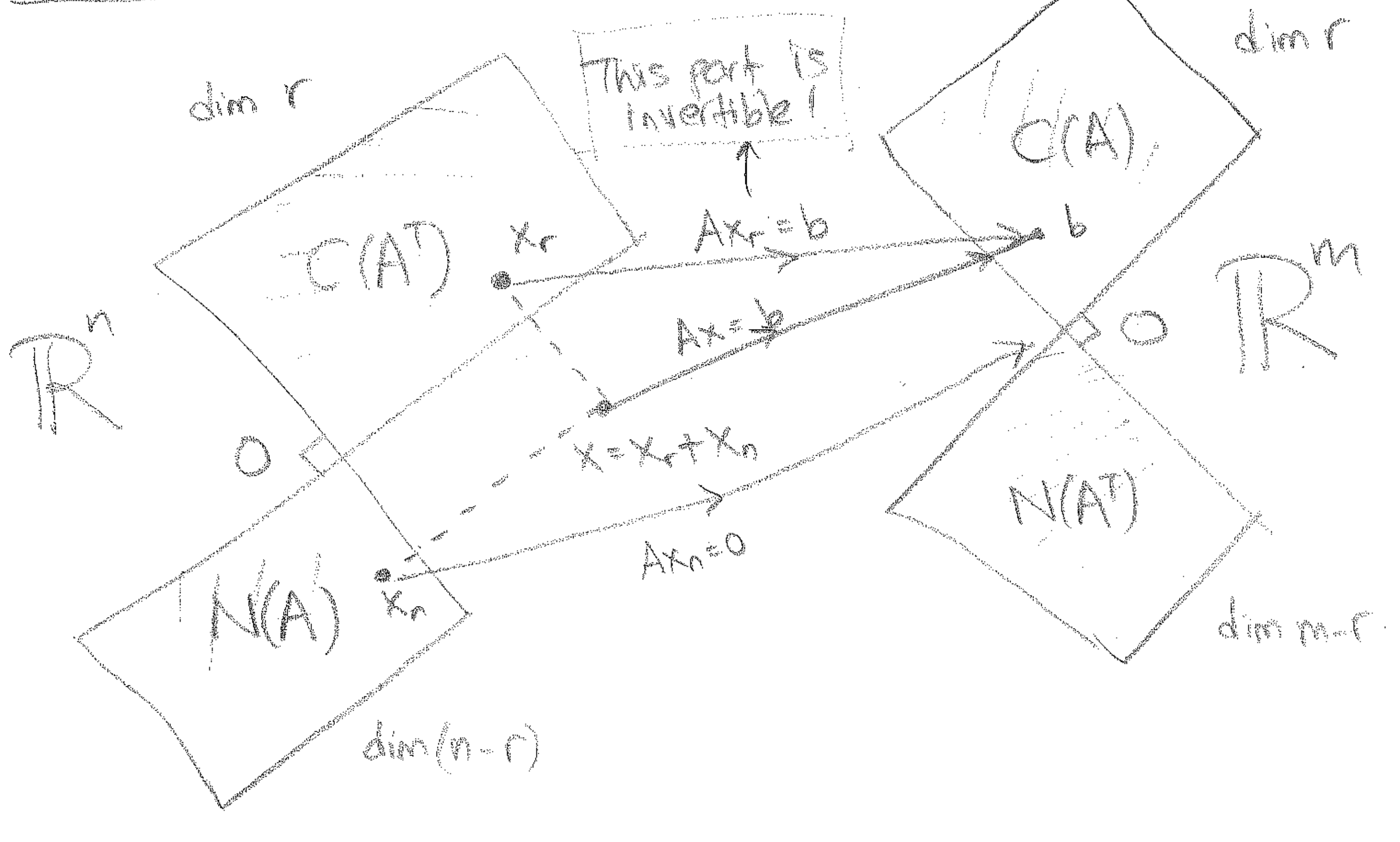
# PROJECTIONS

Last time:

- Orthogonality of vectors: $\vec{v}^T\vec{w} = 0 = \vec{w}^T\vec{v}$.
- Orthogonality of subspaces: $V$ is orth to $W$ if EVERY $\vec{v}$ in $V$ is orth to EVERY $\vec{w}$ in $W$
- Orthogonal complements: $V^\perp = \{$all vectors orthogonal to every vector in $\vec{v}\}$ (perp!)
- A collection of pairwise orthogonal vectors is linearly independent.

$\longrightarrow$ Given any $m\times n$ matrix $A$,

$N(A) = R(A)^\perp$ in $\mathbb{R}^n$
$C(A) = N(A^T)^\perp$ in $\mathbb{R}^m$.



**Thm** Every $\vec{b}$ in $C(A)$ comes from exactly ONE $\vec{x}_r$ in $C(A^T)$, i.e., the equation

$$A\vec{x} = \vec{b}$$

for $\vec{b}$ in $C(A)$ has ONLY ONE solution in $C(A^T)$ [But maybe many in $\mathbb{R}^n$!]

**Pf** If there are TWO solutions, $\vec{x}_1$ & $\vec{x}_2$, [both in $C(A^T)$!]

then: $A\vec{x}_1 = \vec{b} = A\vec{x}_2$,

So $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$,

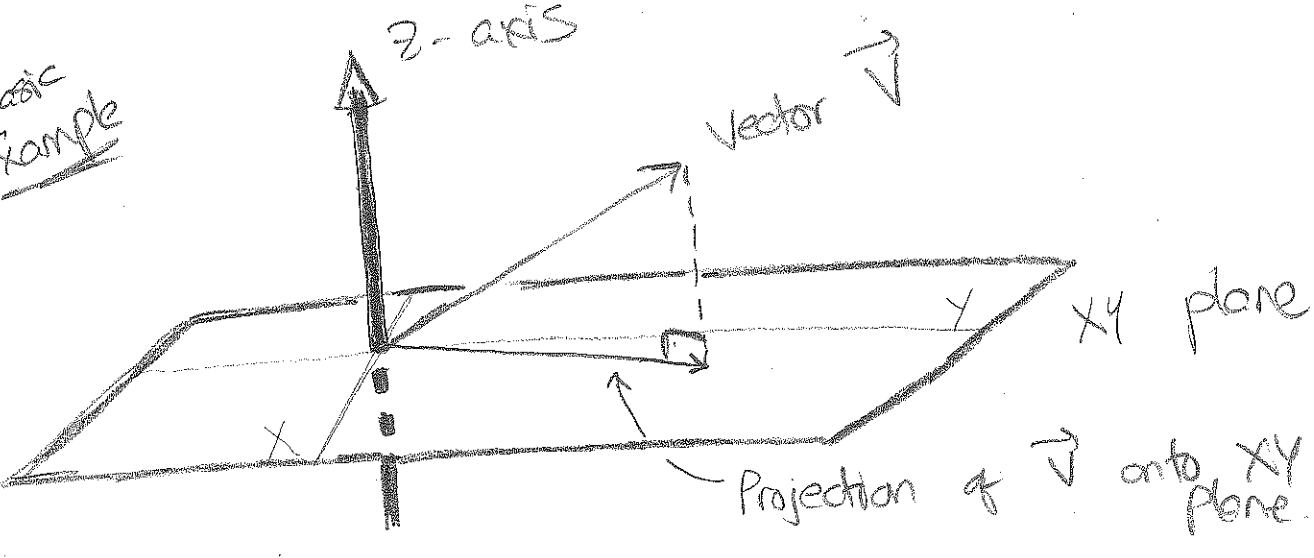
So $\vec{x}_1 - \vec{x}_2$ is in $N(A)$. BUT since $C(A^T)$ is a vector space, and $\vec{x}_1$ & $\vec{x}_2$ are in $C(A^T)$, the linear combination $\vec{x}_1 - \vec{x}_2$ is ALSO in $C(A^T)$.

So: $\vec{x}_1 - \vec{x}_2$ is in $N(A)$ and $C(A^T)$.

Since $N(A)$ and $C(A^T)$ are ORTHOGONAL, we must get $\vec{x}_1 - \vec{x}_2 = \vec{0}$, so ONLY one solution

# PROJECTIONS ONTO SUBSPACES

Basic Example

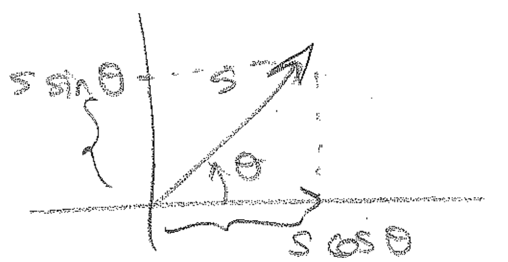


## Why Projections?

Ans: **Physics**. In particular, when using the equations of motion – when an object is launched at speed $s$ m/sec at an angle of $\theta$, then its horizontal/vertical components are:

$s\cos\theta$ (horizontal) $\to$ How far?

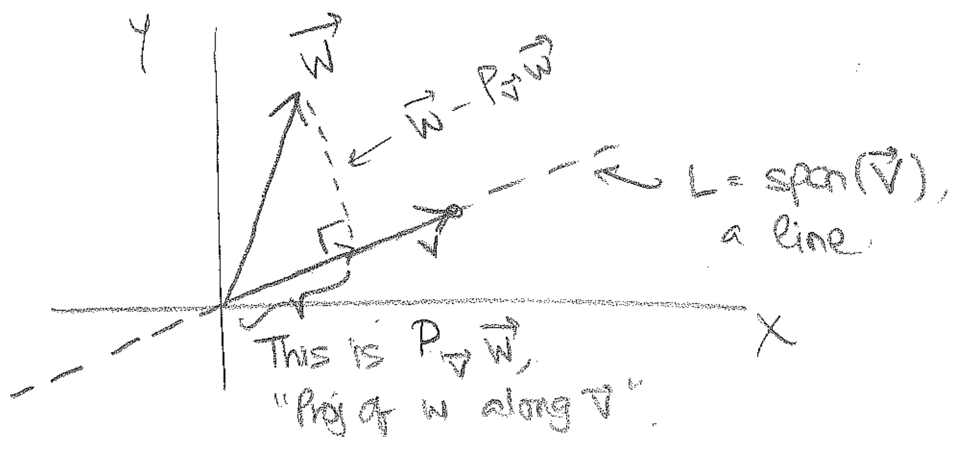
$s\sin\theta$ (vertical) $\to$ How high?




$\theta = 0$, $\cos\theta = 1$, all horizontal.

$\theta = \pi/2$, $\sin\theta = 1$, all vertical!

## PROJECTION ONTO A LINE: (through origin)




Well, projection of $\vec{w}$ along $\vec{v}$ must be SOME multiple of $\vec{v}$!

$$P_{\vec{v}}\vec{w} = \underline{c} \cdot \vec{w}$$

$\to$ Must find this.

Well, we know $(\vec{w} - P_{\vec{v}}\vec{w})$ is perpendicular to $\vec{v}$, so:

$$\vec{v}^{\,T}(\vec{w} - P_{\vec{v}}\vec{w}) = 0$$

which means:

$$\vec{v}^T(\vec{w} - c\vec{v}) = 0$$

or, $\vec{v}^T\vec{w} = c\,\vec{v}^T\vec{v} \rightarrow c = (v^Tv)^{-1}v^Tw.$

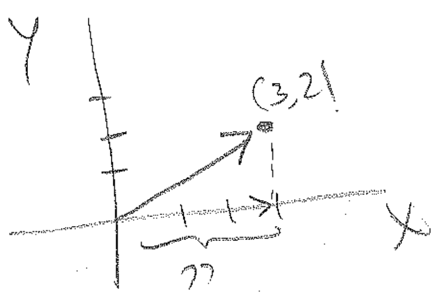
so, $c = \dfrac{\vec{v}^T\vec{w}}{\|\vec{v}\|^2}$ ← (Never zero if $\vec{v} \neq \vec{0}$)

or $= \dfrac{\vec{v}^T\vec{w}}{\vec{v}^T\vec{v}}$

So $$\boxed{P_{\vec{v}}\vec{w} = \overbrace{\left[\frac{\vec{v}^T\cdot\vec{w}}{\vec{v}^T\cdot\vec{v}}\right]}^{c}\vec{v}}$$

CAREFUL, Nothing cancels!

Silly example: What is the projection of (3,2) along the X-axis?




Ans: X-axis = span(1,0). So,

$$P_{\begin{bmatrix}1\\0\end{bmatrix}}\begin{bmatrix}3\\2\end{bmatrix} = \frac{[1\ 0]\begin{bmatrix}3\\2\end{bmatrix}}{[1\ 0]\begin{bmatrix}1\\0\end{bmatrix}}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \frac{3}{1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$ (Duh!!)

NOT so silly:

What is the projection of (3,2,12,7,8,15) along span(1,2,3,4,5,6)??

MUCH LESS SILLY: PROJECTION ONTO ANY SUBSPACE.

Q. Let's say $\vec{u}, \vec{v}$ are the basis for a 2D subspace $V$ of $\mathbb{R}^n$. If $\vec{w}$ is some vector in $\mathbb{R}^n$, what is $P_V\vec{w}$? (Projection of $\vec{w}$ on $V$)

---

Ans We know:

- $P_V\vec{w} = c\vec{u} + d\vec{v}$ (for some c, d that we want to get)

AND - $(\vec{w} - P_V\vec{w})$ is perpendicular to $\vec{u}$ and $\vec{v}$

So, $\vec{u}^T(\vec{w} - c\vec{u} - d\vec{v}) = 0$

and $\vec{v}^T(\vec{w} - c\vec{u} - d\vec{v}) = 0$ } Two equations (Not one!)

Let A be the matrix $[\vec{u} \ \vec{v}]$. Then, our 2 equations become:

$$\left(\begin{array}{l} \vec{u}^T\vec{w} = c\,\vec{u}^T\vec{u} + d\,\vec{u}^T\vec{v} \\ \vec{v}^T\vec{w} = c\,\vec{v}^T\vec{u} + d\,\vec{v}^T\vec{v} \end{array}\right)$$

$$\|$$

$$\left(A^T\vec{w} = A^TA\begin{bmatrix} c \\ d \end{bmatrix}\right).$$

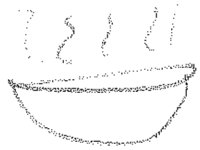
($A^T$ is invertible on its row sp., i.e. on A's col sp)

[ BUT $A^TA$ is a 2×2 matrix of Rank 2 ]

So, $\begin{bmatrix} c \\ d \end{bmatrix} = (A^TA)^{-1}A^T\vec{w}$

Finally, $P_V\vec{w} = A\begin{bmatrix} c \\ d \end{bmatrix} = \left[A(A^TA)^{-1}A^T\right]\vec{w}$

So, $P_V = \underbrace{A(A^TA)^{-1}A^T}_{\text{Matrix which projects onto } V}$

GENERAL RECIPE :  in $\mathbb{R}^n$,

To project a vector $\vec{w}$ onto a subspace $V$, (dim $k$)

- Find a basis $\vec{u}_1, \ldots, \vec{u}_k$ of $V$.
- Set $A = [\vec{u}_1 \ \ldots \ \vec{u}_k]$
- Multiply $\vec{w}$ on the left by:

$$\Big[\underbrace{A^T}_{k\times n \text{ matrix.}} \ \underbrace{(A^T A)^{-1}}_{\text{invertible } k\times k \text{ matrix}} \ \underbrace{A}_{n\times k \text{ matrix}}\Big]$$

---

WARNING Unless $k=n$, we can't use

$$(A^T A)^{-1} = A^{-1}(A^T)^{-1}.$$

Because $A$ and $A^T$ need NOT be invertible themselves

---